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A Minimax Problem Using Generalized Rational Functions*

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I. INTRODUCTION

Let X be a compact metric space and $C(X)$ the space of continuous real-valued functions defined on X. Assume that P and Q are subsets in $C(X)$ and $q(x) > 0$ in X for all $q \in Q$. Then we may construct the function family

$$R = \{p/q \colon p \in P,\ q \in Q\}.$$

We suppose now that $E(z, x)$ is a nonnegative function from $(-\infty, \infty) \times X$ into $[0, \infty)$ such that $\|E(r, \cdot)\| < \infty$ for any element $r \in R$, where

$$\|E(r, \cdot)\| = \sup_{x \in X} E(r, x) \qquad (E(r, x) \equiv E(r(x), x)).$$

Our minimax problem then is to find an element $r_0 \in R$ such that

$$\|E(r_0, \cdot)\| = \inf_{r \in R} \|E(r, \cdot)\|;$$

such an r_0 (if any) is said to be a minimum to E from R.

In this paper we investigate such a problem and study characterization and uniqueness of a minimum to E when P and Q are arbitrary convex sets. Then, as an example, we use these results to deduce the corresponding results for one-sided simultaneous rational approximation.

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II. Characterization and Uniqueness

Suppose P and Q both are convex sets in $C(X)$. Letting $p, p_0 \in P$ and $q, q_0 \in Q$, for $t \in [0, 1]$ write

$$p_t = p_0 + t(p - p_0),$$
$$q_t = q_0 + t(q - q_0),$$
$$r_t = p_t/q_t.$$

Our main results require several lemmas.

Lemma 1. *Let $f(x)$ be a convex function. Then*

$$\phi(t) \equiv \frac{f(r_t) - f(r_0)}{t} \cdot \frac{q_t}{q}$$

is an increasing function of t in $(0, 1]$.

Proof. Since

$$\frac{f(r_t) - f(r_0)}{t} = \frac{f(r_t) - f(r_0)}{r_t - r_0} \cdot \frac{r_t - r_0}{t}$$
$$= \frac{f(r_t) - f(r_0)}{r_t - r_0} \cdot \frac{q}{q_t}(r - r_0),$$

we have

$$\phi(t) = \frac{f(r_t) - f(r_0)}{r_t - r_0}(r - r_0).$$

From

$$r_t - r_0 = \frac{tq}{q_0 + t(q - q_0)}(r - r_0)$$

it follows that in the case $r - r_0 > (<) 0$, for $t \in (0, 1]$ $r_t - r_0 > (<) 0$ and r_t is an increasing (a decreasing) function. By the convexity of f then $(f(r_t) - f(r_0))/(r_t - r_0)$ is an increasing (a decreasing) function of t in $(0, 1]$ [1, p. 6]. Thus in any case we can conclude that $\phi(t)$ is an increasing function of t in $(0, 1]$.

From $\phi(t) \leqslant \phi(1)$ we obtain the following corollary.

COROLLARY. *Let $f(x)$ be a convex function. Then*

$$\frac{f(r_t)-f(r_0)}{t}\cdot\frac{q_t}{q}\leqslant f(r)-f(r_0),\qquad t\in(0,1]. \tag{1}$$

In order to state the following basic lemma we need to introduce the notation

$$X_r=\{x\in X\colon E(r,x)=\|E(r,\cdot)\|\}$$

and to generalize the notion of the directional derivative to be applicable to our case. To this end for $r_i=p_i/q_i$, $i=0,1,2$ denote

$$E'(r_0,x;r_1,r_2)=\lim_{t\to 0+}\left(E\left(\frac{p_0+t(p_1-p_2)}{q_0+t(q_1-q_2)},x\right)-E(r_0,x)\right)\Big/t$$

if the limit exists. Thus

$$E'(r_0,x;r,r_0)=\lim_{t\to 0+}(E(r_t,x)-E(r_0,x))/t.$$

LEMMA 2. *Let P and Q be convex sets in $C(X)$. Suppose that $E(z,x)$ is convex with respect to z for each $x\in X$. Then for any $r,r_0\in R$*

$$\sup_{x\in X_{r_0}}E'(r_0,x;r,r_0)\frac{q_0(x)}{q(x)}\leqslant\|E(r,\cdot)\|-\|E(r_0,\cdot)\|$$
$$\leqslant-\sup_{x\in X_r}E'(r,x;r_0,r)\frac{q(x)}{q_0(x)}. \tag{2}$$

Proof. By the corollary

$$\frac{E(r_t,x)-E(r_0,x)}{t}\cdot\frac{q_t(x)}{q(x)}\leqslant E(r,x)-E(r_0,x).$$

The left expression of the inequality is increasing with respect to t in $(0,1]$ by Lemma 1 and always possesses a limit $E'(r_0,x;r,r_0)\,q_0(x)/q(x)$ as $t\to 0+$. Thus for $x\in X_{r_0}$

$$E'(r_0,x;r,r_0)\,q_0(x)/q(x)\leqslant E(r,x)-E(r_0,x)$$
$$\leqslant\|E(r,\cdot)\|-\|E(r_0,\cdot)\|.$$

Hence

$$\sup_{x\in X_{r_0}}E'(r_0,x;r,r_0)\,q_0(x)/q(x)\leqslant\|E(r,\cdot)\|-\|E(r_0,\cdot)\| \tag{3}$$

which is the left inequality in (2). And the right inequality in (2) follows from interchanging r and r_0 in (3).

THEOREM 1 (Characterization). *Let P and Q be convex sets in $C(X)$ and $r_0 \in R$. Suppose that $E(z, x)$ is convex with respect to z for each $x \in X$ and $E(r, x)$ is upper semicontinuous with respect to x for each $r \in R$. Then r_0 is a minimum to E from R if and only if*

$$\sup_{x \in X_{r_0}} E'(r_0, x; r, r_0) \geqslant 0, \qquad \forall r \in R. \tag{4}$$

Proof. Sufficiency. It directly follows from (2) because $q(x)\, q_0(x) > 0$ in X.

Necessity. Suppose on the contrary that it is possible to find an element $r \in R$ satisfying that

$$\sup_{x \in X_{r_0}} E'(r_0, x; r, r_0) < 0.$$

The remainder of the proof is devoted to showing how to select t, $0 < t \leqslant 1$, so that

$$\|E(r_t, \cdot)\| < e \equiv \|E(r_0, \cdot)\|.$$

First take $y \in X_{r_0}$. We have

$$E'(r_0, y; r, r_0) < 0$$

or

$$\lim_{t \to 0+} (E(r_t, y) - E(r_0, y))/t < 0.$$

So there exists a number $t_y \in (0, 1]$ such that

$$E(r_{t_y}, y) < E(r_0, y),$$

i.e.,

$$E(r_{t_y}, y) < e. \tag{5}$$

By the upper semicontinuity of E with respect to x we may find a neighborhood N_y of the point y such that

$$E(r_{t_y}, x) < e, \qquad \forall x \in N_y. \tag{6}$$

By Lemma 1

$$\frac{E(r_t, x) - E(r_0, x)}{t} \cdot \frac{q_t(x)}{q(x)} \leqslant \frac{E(r_{t_y}, x) - E(r_0, x)}{t_y} \cdot \frac{q_{t_y}(x)}{q(x)}, \quad \forall t \in (0, t_y].$$

Hence for $t \in (0, t_y]$

$$E(r_t, x) \leqslant (tq_{t_y}(x)/t_y q_t(x))\, E(r_{t_y}, x) + (1 - tq_{t_y}(x)/t_y q_t(x))\, E(r_0, x).$$

Since

$$tq_{t_y}/t_y q_t = (tq_0 + tt_y(q - q_0))/(t_y q_0 + tt_y(q - q_0)) \leqslant 1,$$

from (6) it follows that for $t \in (0, t_y]$ and $x \in N_y$

$$E(r_t, x) < (tq_{t_y}(x)/t_y q_t(x))\, e + (1 - tq_{t_y}(x)/t_y q_t(x))\, e = e.$$

This gives that

$$E(r_t, x) < e, \qquad \forall t \in (0, t_y], \quad \forall x \in N_y. \tag{7}$$

Next take $y \in X \backslash X_{r_0}$. We have $E(r_0, y) < e$. Since by the corollary

$$\lim_{t \to 0+} E(r_t, y) \leqslant E(r_0, y),$$

we can find a positive number t_y such that (5) is also valid. The same argument as above may also give (7).

Now from the open cover $\{N_y\}$ of the compact metric space X we may select a finite subcover $\{N_{y_1}, \dots, N_{y_n}\}$. Taking the minimum of the corresponding numbers $t_{y_1}, \dots, t_{y_n}$, denoted by t, then we obtain that $0 < t \leqslant 1$ and

$$E(r_t, x) < e, \qquad \forall x \in X$$

Whence

$$\|E(r_t, \cdot)\| < e.$$

We have reached a contradiction because $r_t = p_t/q_t \in R$.

THEOREM 2 (Characterization). *Under the assumptions of Theorem* 1 *if E possesses a minimum from R, then r_0 is a minimum to E from R if and only if*

$$\sup_{x \in X_r} E'(r, x; r_0, r) \leqslant \sup_{x \in X_{r_0}} E'(r_0, x; r, r_0), \qquad \forall r \in R. \tag{8}$$

Proof. If r_0 is a minimum to E from R, i.e.,

$$\|E(r, \cdot)\| \geqslant \|E(r_0, \cdot)\|, \qquad \forall r \in R,$$

then

$$\sup_{x \in X_r} E'(r, x; r_0, r) \leqslant 0, \qquad \forall r \in R$$

by Lemma 2 and (4) is valid by Theorem 1. So (8) follows.

Conversely, let r_0 satisfy (8). Suppose on the contrary that r_0 is not a minimum to E from R but $r \in R \backslash \{r_0\}$ is. Thus by Lemma 2

$$\sup_{x \in X_{r_0}} E'(r_0, x; r, r_0) < 0$$

and by Theorem 1

$$\sup_{x \in X_r} E'(r, x; r_0, r) \geqslant 0. \tag{9}$$

This is a contradiction.

THEOREM 3 (Uniqueness). *Under the assumptions of Theorem* 2 *the following statements are equivalent to each other*:

(a) $\|E(r_0, \cdot)\| < \|E(r, \cdot)\|$, $\forall r \in R \backslash \{r_0\}$;

(b) $\sup_{x \in X_r} E'(r, x; r_0, r) < 0$, $\forall r \in R \backslash \{r_0\}$;

(c) $\sup_{x \in X_r} E'(r, x; r_0, r) < \sup_{x \in X_{r_0}} E'(r_0, x; r, r_0)$, $\forall r \in R \backslash \{r_0\}$.

Proof. (a) $\Rightarrow$ (b) By Lemma 2 it directly follows.

(a) $\Rightarrow$ (c) Since (a) implies (4) by Theorem 1, (c) follows from (4) and (b).

(b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) Suppose not and let $r \in R \backslash \{r_0\}$ be a minimum to E from R. Then by Theorem 1 inequality (9) is valid and by Lemma 2

$$\sup_{x \in X_{r_0}} E'(r_0, x; r, r_0) \leqslant 0. \tag{10}$$

But (9) contradicts (b), and (9) and (10) together contradict (c).

Remark. The all results of this section remain true if we take $R^* \subset R$ instead of R, provided that R^* satisfies the condition:

$$r, r_0 \in R^* \Rightarrow r_t \in R^*, \qquad t \in (0, 1). \tag{11}$$

III. Application

Let

$$E(z, x) = \sum_j \lambda_j |f_j(x) - z|, \qquad f_j \in C(X), \quad \lambda_j \geqslant 0 \quad \text{and} \quad \sum_j \lambda_j = 1,$$

and let

$$R^* = \{r \in R\colon r(x) \leqslant \inf_j \{f_j(x)\}\}.$$

This is a one-sided simultaneous rational approximation problem. It is easy to see that the set R^* satisfies condition (11).

Putting $r, r_0 \in R^*$ we have

$$\begin{aligned} E'(r_0, x; r, r_0) &= \lim_{t \to 0+} (E(r_t, x) - E(r_0, x))/t \\ &= \lim_{t \to 0+} \sum_j \lambda_j(|f_j(x) - r_t(x)| - |f_j(x) - r_0(x)|)/t \\ &= \lim_{t \to 0+} \sum_j \lambda_j(r_0(x) - r_t(x))/t \\ &= (r_0(x) - r(x))\, q(x)/q_0(x). \end{aligned}$$

Thus (4) becomes

$$\sup_{x \in X_{r_0}} (r_0(x) - r(x)) \geqslant 0, \qquad \forall r \in R^*. \tag{12}$$

Similarly, (b) in Theorem 3 becomes

$$\sup_{x \in X_r} (r(x) - r_0(x)) < 0, \qquad \forall r \in R^* \backslash \{r_0\}. \tag{13}$$

By the remark above, a corollary to Theorem 1 and Theorem 3 follows:

Corollary 1. *Let P and Q be convex sets in $C(X)$. An element $r_0 \in R^*$ is a best approximation to $\{f_j\}$ from R^* if and only if* (12) *is valid. Meanwhile, if $\{f_j\}$ possesses a best approximation from R^*, then $r_0 \in R^*$ is the unique best approximation to $\{f_j\}$ from R^* if and only if* (13) *is valid.*

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Reference

1. A. W. Roberts and D. E. Varberg, "Convex Functions," Academic Press, New York/London, 1973.